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ON THE PARTIAL STABILITY AND CONVERGENCE OF MOTIONS*

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A sufficient condition of stability of the zero solution of a system of ordinary differential equations with respect to a part of variables, and for these variables to approach constant values over prolonged time, is obtained using the modified second Liapunov's method. The results are applied in the investigation of conditions of asymptotic approach of motions of unsteady mechanical systems to one of the equilibrium positions in the presence of dry friction. A symmetric gyroscope suspended in gimbals is used as an example.

It is often observed in mechanical systems subjected to dissipative forces that the generalized coordinates (or part of them) approach along motions constant values and the generalized velocities approach zero after long time intervals, i.e. the system tends to "stop asymptotically" /1—5/.

Asymptotic stability and asymptotic stability relative to a part of variables /1,2/ are particular cases of this effect which results in the generalized coordinates approaching those of the equilibrium position. A sufficient condition of existence of this phenomenon is given below. The result is a generalization and further development of several previous investigations /6,7/, and makes possible a simple derivation of the Liapunov—Malkin theorem /1,8/ on the existence of a limit of the part of coordinates, and obtain its extension to nonautonomous systems using the second Liapunov's method. Its application in investigations of holonomic mechanical systems subjected to potential, gyroscopic, and dissipative forces /3-5/ shows that the "asymptotic stopping" is a particular characteristic of the dry friction effect /9/.

1. Consider the system of differential equations

$$x^{\cdot} = X(t, x), \quad X(t, 0) \equiv 0, \quad x = (y_1, \ldots, y_p, z_1, \ldots, z_q)^T \qquad (1.1)$$
$$n \geqslant p > 0, \quad p + q = n$$

where x is a real n-vector. Introducing the notation $y = (y_1, \ldots, y_p)^T$, $z = (z_1, \ldots, z_q)^T$, we reduce system (1.1) to the form

$$y^{\cdot} = Y(t, x), \quad z^{\cdot} = Z(t, x) \qquad (1.2)$$

We denote

$$\|y\| = \Big(\sum_{i=1}^{p} y_i^2\Big)^{1/2}, \quad \|z\| = \Big(\sum_{i=1}^{q} z_i^2\Big)^{1/2}, \quad \|x\| = (\|y\|^2 + \|z\|^2)^{1/2}$$

Assume the vector functions Y and Z to be continuous in the domain $t \in R_+ = [0, \infty)$, $\|y\| < H$, $0 < \|z\| < \infty$ and such that solutions $(y(t; t_0, x_0), z(t; t_0, x_0))$ of system (1.2) continuously depend on initial data $x_0 = (y(t_0; t_0, x_0), z(t_0; t_0, x_0))$ and are z-continuable /2,10/. When the continuous function $V: \Gamma_y \to R$ satisfies the local Lipschitz condition with respect to x in the set

$$\Gamma_y = \{(t, x): t \in R_+, \|y\| < H', 0 \leqslant \|z\| < \infty\} \ (0 < H' < H)$$

we call, by virtue of system (1.1) /11/, function

$$V^{\cdot}(t, x) = \limsup_{h \to +0} (V(t + h, x + hX(t, x)) - V(t, x))/h$$

the derivative of function V.

We denote by K the class of strongly increasing functions $a: R_+ \to R_+$, for which $a(0) = 0$.

Theorem. Assume that there exists for system (1.2) a continuous function $V: \Gamma_y \to R$ which satisfies the local Lipschitz condition with respect to x and, also, the following

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conditions on the set Γ_y:

1) $V(t, x) \geqslant 0$;

2) there exists functions

$$\omega \in K \quad \text{and} \quad w: R_+ \to R_+\left(\int_0^\infty w < \infty\right)$$

such that

$$V^{\cdot}(t, x) \leqslant -\omega(\|y\|)\|Y(t, x)\| + w(t)$$

Then

a) if $V(t, 0) \to 0$ as $t \to \infty$, the zero solution of system (1.2) is y-stable;

b) for any solution $x(t) = (y(t), z(t))^T$ of system (1.2) for which $\|y(t)\| \leqslant H'$ when $t \geqslant t_0$ function $y(t)$ has a finite limit as $t \to \infty$.

Proof. Let $x(t)$ be a solution of (1.2), $0 < \gamma' < \gamma'' < H'$. Assume that there exist $T, t', t'', k (T \leqslant t' < t'', 1 \leqslant k \leqslant p)$, such that $|y_k(t')| = \gamma'$, $|y_k(t'')| = \gamma''$. Then denoting $v(t) = V(t, x(t))$, we obtain

$$v(t') - v(t'') \geqslant -\int_{t'}^{t''} V^{\cdot}(t, x(t))\, dt \geqslant -\int_T^{t''} w(t)\, dt + \int_{t'}^{t''} \omega(|y_k(t)|)|Y_k(t, x(t))|\, dt \geqslant -\int_T^{t''} w(s)\, ds + \int_{\gamma'}^{\gamma''} w(\tau)\, d\tau = I(T, t'', \gamma', \gamma'') \tag{1.3}$$

a) let $t_0 \in R_+$ and $\varepsilon > 0$ be specified. By virtue of conditions

$$V(t, 0) \to 0 \quad (t \to \infty) \quad \text{and} \quad \int_0^\infty w < \infty$$

there exist $\delta_1 (0 < \delta_1 < \varepsilon/(2p))$ and $T > t_0$, such that $\|x\| < \delta_1$ and $T < t''$ imply the existence of $V(T, x) < I(T, t'', \varepsilon/(2p), \varepsilon/p)$. Since solutions are continuously dependent on initial data, there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that for $\|x_0\| < \delta$ the inequality $\|x(t; t_0, x_0)\| < \delta_1$ is satisfied in the interval $[t_0, T]$. Let us prove that when $t \geqslant t_0$ we have $\|y(t; t_0, x_0)\| < \varepsilon$, i.e. zero solution of system (1.2) is y-stable. Otherwise there would exist numbers t', t'', k $(T < t' < t'', 1 \leqslant k \leqslant p)$ such that $|y_k(t')| = \varepsilon/(2p)$, $|y_k(t'')| = \varepsilon/p$. Then by virtue of (1.3)

$$V(T, x(T)) = v(T) \geqslant -\int_T^{t''} v^{\cdot}(t)\, dt \geqslant I\left(T, t'', \frac{\varepsilon}{2p}, \frac{\varepsilon}{p}\right)$$

but this contradicts the definition of δ_1.

b) If the theorem is false, there exist numbers γ', γ'', k $(0 < \gamma' < \gamma'', 1 \leqslant k \leqslant p)$ and sequencies $\{t_i'\}, \{t_i''\}$ such that $t_0 < t_i' < t_i'' < t_{i+1}'$, $|y_k(t_i')| = \gamma'$, $|y_k(t_i'')| = \gamma''$ $(i = 1, 2, \ldots)$, and then by virtue of (1.3)

$$v(t_i'') \leqslant v(t_0) - i\int_{\gamma'}^{\gamma''} \omega(\tau)\, d\tau + \int_{t_0}^{t_i''} w(s)\, ds \to -\infty \quad (i \to \infty)$$

which contradicts condition 1).

This theorem can also be used for deriving the conditions of asymptotic y-stability.

Definition /12/. The zero solution of system (1.2) has the property of weak attraction with respect to y, if for any $t_0 \in R_+$ there exists a $\sigma > 0$ such that

$$\liminf_{t \to \infty} \|y(t; t_0, x_0)\| = 0 \quad \text{for} \quad \|x_0\| < \sigma$$

Corollary 1.1. If the conditions of Theorem 1.1 are satisfied and $V(t, 0) \to 0$ $(t \to \infty)$, the property of weak attraction with respect to y has as a consequence the y-stability of the zero solution.

Corollary 1.2. If there exists a nonnegative function $V: \Gamma_y \to R$ $(V(t, 0) \equiv 0)$ whose derivative by virtue of system (1.2) is negative definite with respect to y and function Y is bounded on set Γ_y, the zero solution of system (1.2) is asymptotically y-stable.

Proof. Function $V^{\cdot}(t,x)$ is negative definite with respect to y, hence there exists a function $c \in K$ such that $V^{\cdot}(t,x) \leqslant -c(\|y\|)$. Let $\|Y(t,x)\| \leqslant M\ ((t,x) \in \Gamma_y)$. Then $V^{\cdot}(t,x) \leqslant -(c(\|y\|)/M)\|Y(t,x)\|$ and the conditions of Theorem 1.1 are satisfied. Since, on the other hand, $V \geqslant 0$ and function $V^{\cdot}$ is negative definite with respect to y, the zero solution has the property of weak attraction with respect to y.

Remark 1.1. Corollary 1.2 differs from the generalization of Marachkov's theorem for partial stability /10/ only in that the condition of positive definiteness of function V has not been stipulated. It should be, however, pointed out that in conformity with the method in /13,14/ Corollary 1.2 implies the definite positiveness of function V with respect to y.

By corollary 1.2 the property of attraction is ensured by the negative definiteness of function $V^{\cdot}$. In the following statement we formulate the sufficient condition for the property of attraction, using function Y.

Lemma. If there exist continuous functions $a, b: R_+ \to R$ and $\tau_0 > 0$ such that

$$b(\tau) > 0\ (\tau > 0),\ b(0) = 0,\ \int_1^H \frac{1}{b(\tau)}\,d\tau = \infty, \qquad 2\liminf_{t\to\infty} \int_{t_0}^t a(s)\,ds \leqslant \int_{\tau_0}^0 \frac{1}{b(\tau)}\,d\tau$$

for all $t_0 \geqslant 0$ and

$$y^T Y(t,x) \leqslant a(t)\, b(\|y\|^2)\ \ (t \in R_+,\ \|y\| \leqslant H) \tag{1.4}$$

the zero solution has the property of attraction with respect to y.

Proof. It follows from (1.4) that for any solution of system (1.2)

$$d(\|y(t)\|^2)/dt \leqq 2y^T(t) Y(t, x, (t)) \leqslant 2a(t) b(\|y(t)\|^2)$$

By virtue of properties of functions a and b the maximum solution of the Cauchy problem $[\tau^{\cdot} = 2a(t)b(\tau);\ \tau(t_0) = \tau_0]$ is determined for $t \geqslant t_0$, and $\liminf_{t\to\infty} \tau^*(t) = 0$ /11/. On the other hand, by the basic theorem of the theory of differential inequalities /11/ $\|y(t)\|^2 \leqslant \tau^*(t)$ when $\|y(t_0)\|^2 < \tau_0$, hence solution $x(t)$ is determined when $t \geqslant t_0$ and $\liminf_{t\to\infty} \|y(t)\| = 0$, i.e. the zero solution has the property of weak attraction with respect to y.

2. Consider the system

$$y^{\cdot} = Y(t,x),\quad z^{\cdot} = A(t)z + Z(t,x) \tag{2.1}$$

where x, y, z, Y, Z are the same as in (1.2), and A is a continuous matrix function $q \times q$ determinate on R_+. Let us assume the existence of constants $c > 0,\ 0 < \gamma \leqq 1$, such that on the set Γ_y

$$\|Y(t,x)\| \leqslant c\|z\|^\gamma,\quad \|Z(t,x)\| = o(\|z\|)\ \ (x \to 0) \tag{2.2}$$

The equations of motion of many mechanical systems can be reduced to form (2.1) /3−5/. Liapunov has shown /1/ that, if matrix A is constant, its eigenvalues have negative real parts and $\gamma = 1$, the zero solution of system (2.1) is then stable, and function $y(t)$ has a finite limit as $t \to \infty$. That theorem was extended in /15/ to the case when matrix A depends on t and $0 < \gamma \leqslant 1$, but proof was only given with additional constraints on function $A(t)$. Using the theorem proved above, the Liapunov method can be readily used in the case of any arbitrary matrix function $A(t)$.

Corollary 2.1. If the zero solution

$$u^{\cdot} = A(t)u \tag{2.3}$$

of the system is exponentially asymptotically stable, the zero solution of system (2.1) is stable, asymptotically z-stable, and for fairly small initial values of $\|x(t_0)\|$ function $y(t)$ has a finite limit as $t \to \infty$.

Proof. Since the zero solution of system (2.3) is exponentially asymptotically stable, there exists a continuous function $V: R_+ \times R^q \to R$ such that

$$\alpha_1\|z\| \leqslant V(t,z) \leqslant \alpha_2\|z\|,\quad |V(t,z) - V(t,z')| \leqslant \alpha_3\|z - z'\|,\quad V^{\cdot}_{(2.3)}(t,z) \leqslant -\alpha_4\|z\|$$

By virtue of (2.2) there exist constants $\alpha_5 > 0$ and $H''\ (0 < H'' < H')$, such that

$$V^{\cdot}_{(2.1)}(t, x) \leqslant V^{\cdot}_{(2.3)}(t, z) + \alpha_3 \| z(t, x) \| \leqslant -\alpha_5 \| z \| \tag{2.4}$$

when $\|x\| < H''$. Function $W(t, x) = V^{\gamma}(t, z) + \varepsilon \|y\|$ with fixed number $\varepsilon\ (0 < \varepsilon < \gamma\alpha_5\alpha_2^{\gamma-1}/c)$ satisfies the local Lipschitz condition for $\|z\| > 0$, and

$$W^{\cdot}_{(2.1)}(t, x) \leqslant \gamma V^{\gamma-1}(t, z) V^{\cdot}_{(2.1)}(t, x) + \varepsilon \| Y(t, x) \| \leqslant -\alpha_6 \| Y(t, x) \| \ (\alpha_6 = -\varepsilon + \gamma\alpha_5\alpha_2^{\gamma-1}/c > 0) \tag{2.5}$$

for $\|x\| \leqslant H''$, $\|z\| > 0$. If for the solution $(y(t), z(t))^T$ of system (2.1) $z(T) = 0$, then $z(t) \equiv 0$ and because of condition (2.2) $y(t) \equiv y(T)$ for $t \geqslant T$. Hence it is sufficient to investigate only such solutions for which $\|z(t)\| > 0$ for $t \geqslant t_0$. By virtue of properties of functions V, W and estimates (2.4) and (2.5) the zero solution of system (2.1) is stable with respect to x and asymptotically z-stable /10/. Consequently, among solutions with fairly small initial values, inequality (2.5) is satisfied. This shows that the theorem in Sect.1 is applicable to system (2.1) and function W, which implies the existence of a limit of function $y(t)$ as $t \to \infty$.

Remark 2.1. When $\gamma = 1$, then $A(t)$ is bounded and $Y(t, x)$ is continuously differentiable, and stability and asymptotic z-stability are implied by Thoerem 3 in /16/, where stability properties of the zero solution of system of the type of (2.1) were investigated in the first approximation under rather general conditions imposed on function $Y(t, x)$.

3. Consider the holonomic mechanical system with time-dependent constraints, defined by the Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial q^{\cdot}} - \frac{\partial L}{\partial q} = Q, \quad L = L_2 + L_1 + L_0 \tag{3.1}$$
$$L_2 = {}^1/_2 (q^{\cdot})^T A(t, q) q^{\cdot}, \ L_1 = B(q)^T q^{\cdot}, \ L_0 = L_0(t, q)$$

where $q \in R_n$ is the vector of generalized coordinates, A, B, L_0 have continuous partial derivatives, and there exists constant $\alpha > 0$ such that

$$\alpha \| q^{\cdot} \|^2 \leqslant (q^{\cdot})^T A(t, q) q^{\cdot} \ ((t, q, q^{\cdot}) \in \Gamma_q \subset R^{2n+1})$$

The system is subjected to dissipative and gyroscopic forces whose resultant is denoted by $Q = Q(t, q, q^{\cdot})$, hence $Q^T q^{\cdot} \leqslant 0$. Let $q = q^{\cdot} = 0$ be the equilibrium position of system (3.1).

Corollary 3.1. Assume that there exists a function $\omega \in K$ such that the inequality

$$Q^T(t, q, q^{\cdot})\ q^{\cdot} \leqslant -\omega(\| q \|) \| q^{\cdot} \| \tag{3.2}$$

is satisfied, and that in space $(t, q, q^{\cdot}) \in R^{2n+1}$ on set Γ_q $L_0 \leqslant 0$, $\partial L / \partial t \geqslant 0$.

Then

a) if $L_0(t, 0) \equiv 0$ when $t \geqslant 0$, the equilibrium position $q = q^{\cdot} = 0$ of system (3.1) is stable;

b) any motion $q(t)$ of system (3.1) for which $\| q(t) \| \leqslant H'$ when $t \geqslant t_0$ has a finite limit as $t \to \infty$.

Proof. Matrix A is positive definite, hence system (3.1) can be represented in the form

$$\frac{d}{dt} q = q^{\cdot}, \quad \frac{d}{dt} q^{\cdot} = F(t, q, q^{\cdot}) \tag{3.3}$$

It was shown in /17/ that

$$\frac{d}{dt}(L_2 - L_0) = -\frac{\partial L}{\partial t} + Q^T q^{\cdot} \tag{3.4}$$

Consequently the derivative of Liapunov is function $V = L_2 - L_0$ satisfies by virtue of system (3.1) the inequality $V^{\cdot}(t, q, q^{\cdot}) \leqslant -\omega(\| q^{\cdot} \|) \| q^{\cdot} \|$. Since function V is positive definite relative to $q^{\cdot}$ and $V^{\cdot} \leqslant 0$, the equilibrium position $q = q^{\cdot} = 0$ is $q^{\cdot}$-stable /10/. Since, however, the conditions of the theorem in Sect. 1 are satisfied by system (3.3) and V, the equilibrium position is q-stable and $q(t)$ has a finite limit as $t \to \infty$.

Note that condition (3.2) excludes the continuity of function Q. However it can be seen that the statements of the theorem in Sect.1 remain valid without the condition of continuity of function X if one assumes that: a) function $X(t, x)$ is measurable in the domain Γ_y and for any bounded closed domain $D \subset \Gamma_V$ there exists a summable function $\tau(t)$ such that almost everywhere in D $\|X(t, x)\| \leqslant \tau(t)$; b) the solution of system (1.1) is understood in the sense given in /18/, i.e. the vector function $x(t)$ determinate on the interval (t_1, t_2) is taken as the solution of Eq. (1.1), if it is absolutely continuous and if for almost all $t \in (t_1, t_2)$ and any $\delta > 0$ vector $dx(t)/dt$ belongs to the smallest convex closed set containing all values of vector function $X(t, x')$ when x' runs through almost the entire δ-neighborhood of point $x(t)$ in space x (at fixed t); c) function $V(t, x)$ is continuously differentiable, and d) $w(t) \equiv 0$, $V(t, 0) \equiv 0$. In such case the condition of continuous dependence of solutions on initial data is not used in the proof of the theorem in Sect.1. System (3.3) and estimate (3.2) obviously satisfy these conditions.

Conditions $L_0 \leqslant 0$ and $L_0(t, 0) = 0$ taken together imply that the potential energy has a minimum when $q = 0$ for all $t \geqslant 0$. The Lagrange function has the property that $\partial L/\partial t \geqslant 0$, if, for instance, for any fixed generalized coordinates and velocities, the kinetic and potential energies are, respectively increasing and decreasing functions of t. Condition (3.2) is satisfied by the dry friction forces /9/

$$Q = \begin{cases} -cq^{\cdot}/\|q^{\cdot}\|, & q^{\cdot} \neq 0 \\ T(t, q), & q^{\cdot} = 0 \end{cases}, \quad Q_i = \begin{cases} -F_i \operatorname{sign} q_i^{\cdot}, & q_i^{\cdot} \neq 0 \\ T_i(t, q, q^{\cdot}), & q_i^{\cdot} = 0 \end{cases}$$
$$(0 < c = \text{const}, \ 0 < F_i = \text{const}, \ i = 1, \ldots, n)$$

4. Let us consider the motion of a symmetric gyroscope in gimbals. Assume that the stationary axis of the external gimbal ring rotation is vertical, and that of the inner ring is horizontal. Let the center of mass of the gyroscope and inner gimble ring be located on the gyroscope axis of symmetry. The position of this system can be defined by the three Euler angles, viz. of nutation θ, precession ψ, and the gyroscope proper rotation φ /19/. We assume that besides forces of gravity the gyroscope is subjected to friction forces at the gimbal ring axes.

The sufficient condition of asymptotic stability of vertical rotations $\theta = \theta^{\cdot} = 0, \psi^{\cdot} = \text{const}$, $\varphi^{\cdot} = \text{const}$ in the presence of viscous friction with total dissipation appeared in /19/. The gyroscope motion was investigated in /20/ in the presence of dry friction forces whose moments about the gimbal ring axes are defined by formulas

$$M_1 = \begin{cases} -D_1 \operatorname{sign} \theta^{\cdot}, & \theta^{\cdot} \neq 0 \\ D_1 \operatorname{sign} f_1(\psi^{\cdot}, \theta), & \theta^{\cdot} = 0, \ |f_1(\psi^{\cdot}, \theta)| > D_1 \\ f_1(\psi^{\cdot}, \theta), & \theta^{\cdot} = 0, \ |f_1(\psi^{\cdot}, \theta)| \leqslant D_1 \end{cases}$$
$$M_2 = \begin{cases} -D_2 \operatorname{sign} \psi^{\cdot}, & \psi^{\cdot} \neq 0 \\ D_2 \operatorname{sign} f_2(\theta^{\cdot}, \theta), & \psi^{\cdot} = 0, \ |f_2(\theta^{\cdot}, \theta)| > D_2 \\ f_2(\theta^{\cdot}, \theta), & \psi^{\cdot} = 0, \ |f_2(\theta^{\cdot}, \theta)| \leqslant D_2 \end{cases}$$
$$f_1(\psi^{\cdot}, \theta) = -(A + B_1 - C_1)\psi^{\cdot 2} \sin\theta \cos\theta + C\tau_0 \psi^{\cdot} \sin\theta - S z_0 \sin\theta$$
$$f_2(\theta^{\cdot}, \theta) = -C\tau_0 \theta^{\cdot} \sin\theta, \quad \tau_0 = \varphi^{\cdot} + \psi^{\cdot} \cos\theta$$

where D_1, D_2 are positive constants, S is the mass of the gyroscope (rotor) and inner gimbal ring, A, A, C are the gyroscope principal moment of inertia, and A_1, B_1, C_1 are the principal moments of inertia of the inner gimbal ring about the axes of the coordinate system rigidly attached to that ring. It will be readily seen (see /20/) that such friction forces at fairly small $|\theta_0|$ admit the motion $\theta = \theta_0 = \text{const}, \theta^{\cdot} = \psi^{\cdot} = 0, \varphi^{\cdot} = \text{const}$, that represents permanent rotation about an axis at angle $\theta = \theta_0$ to the vertical. It was shown in /20/ that when in the position $\theta = 0$ the center of mass of the gyroscope and inner gimbal is under the common center of gimbals, these motions are stable.

Let us investigate the conditions of "asymptotic stopping" of a gyroscope. Disregarding the ignorable coordinate, the equations of motion are of the form (where R is the Routh function /19/)

$$\frac{d}{dt}\frac{\partial R}{\partial \theta^{\cdot}} - \frac{\partial R}{\partial \theta} = -\frac{\partial P}{\partial \theta} + M_1 \tag{4.1}$$
$$\frac{d}{dt}\frac{\partial R}{\partial \psi^{\cdot}} - \frac{\partial R}{\partial \psi} = M_2$$
$$2R = A(\theta^{\cdot 2} + \psi^{\cdot 2} \sin^2\theta) + A_1 \theta^{\cdot 2} + B_1 \psi^{\cdot 2} \sin^2\theta + C_1 \psi^{\cdot 2} \cos^2\theta + A_2 \Psi^{\cdot 2}, \ P = S z_0 (\cos\theta + 1)$$

It is know that

$$(R + P)^{\cdot} = M_1 \theta^{\cdot} + M_2 \Psi^{\cdot} \leqslant -D_1 |\theta^{\cdot}| - D_2 |\Psi^{\cdot}|$$

Moreover function $R+P$ is bounded below along the motions, hence $\int^{\infty}|\theta^{\cdot}|<\infty, \int^{\infty}|\psi^{\cdot}|<\infty$. We can state that $\theta^{\cdot}(t)\to 0$, $\psi^{\cdot}(t)\to 0$ as $t\to\infty$. Let us assume, for example, that $\theta^{\cdot}(t)$ approaches zero as $t\to\infty$. There exist then $t_k'<t_k''<t_{k+1}'$ $(k=1,2,\ldots)$ and $\delta>0$ such that as $t_k''-t_k'\to 0$

$$\left|\int_{t_k'}^{t_k''}\theta^{\cdot\cdot}(t)\,dt\right|=|\theta^{\cdot}(t_k'')-\theta^{\cdot}(t_k')|=\delta\quad(k\to\infty)$$

which contradicts the boundedness of $\theta^{\cdot\cdot}(t_0)$.

On the other hand system (4.1) satisfies the conditions of statement b) of Corollary 2.2, hence $\theta(t)\to\mathrm{const}$, $\Psi(t)\to\mathrm{const}$, and because of the ignorable integral $\varphi^{\cdot}+\psi^{\cdot}\cos\theta=\tau_0$ function $\varphi^{\cdot}(t)\to\mathrm{const}$ as $t\to\infty$.

The above proves the following statement: when dry friction forces with moments M_1 and M_2 act on the gimbal ring axes, then for any initial conditions $\theta(t)\to\mathrm{const}, \psi(t)\to\mathrm{const}$, $\theta^{\cdot}(t)$, $\Psi^{\cdot\cdot}(t)\to 0$, $\varphi^{\cdot}(t)\to\mathrm{const}$ as $t\to\infty$, i.e. every motion asymptotically approaches one of the permanent rotations $\theta=\mathrm{const}$, $\theta^{\cdot}=\psi^{\cdot}=0$, $\varphi^{\cdot}=\mathrm{const}$.

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